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Redesigning the Dynamics of Structural Systems

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The critical link between two powerful modeling techniques, component mode synthesis and large admissible perturbation theory, is established. This new redesign process allows structural systems that are composed of several substructures to be redesigned. The process is developed in three steps. First, the analytical first-order perturbation relationships between the structural system eigenvalues and the eigenvalues of each independent substructure are developed. Second, a transformation matrix is developed for each perturbation relationship, then combined to form the ultimate structural transformation matrix. Third, a minimization problem is defined that uses the ultimate transformation matrix as a set of constraint equations. The result of this minimization is a set of changes to the substructure eigenvalues that approximately affect the objective changes to the structural system eigenvalues. The success of the redesign process is demonstrated by an isolated platform example. The changes in the substructure eigenvalues are predicted via the redesign process, then substituted back into the full nonlinear equations for the structural system, and the results are discussed. This work forms the basis for future redesign developments when other static and dynamic objectives are to be achieved for structural systems.

Nomenclature

$[T_{X,Y}]$ = transformation matrix from the $\{\Delta Y\}$ space to the $\{\Delta X\}$ space
$[X]$ = matrix of baseline structure
$[X]$ = $[\{x_1\}\{x_2\}\cdots]$
$\{X\}$ = $\begin{Bmatrix} \{x_1\} \\ \{x_2\} \\ \vdots \end{Bmatrix}$
$[X']$ = matrix of objective structure
$[\Delta X]$ = matrix change from the baseline to the objective structure, $[X'] = [X] + [\Delta X]$
$[\Delta X]$ = $[\{\Delta x_1\}\{\Delta x_2\}\cdots]$
$\{\Delta X\}$ = $\begin{Bmatrix} \{\Delta x_1\} \\ \{\Delta x_2\} \\ \vdots \end{Bmatrix}$

I. Introduction

STRUCTURAL models generally are developed to predict static and dynamic responses of a structure. These responses are then assessed with respect to the objective performance criteria of the structure. When the predicted responses do not meet the performance criteria, the model may be used to predict the design changes that will affect the objective changes in performance. This is the redesign process. A new redesign methodology is developed in this work that is applicable to a structural system model that is composed of many substructure models.

Currently the ability to make large changes in the performance of a single structure is achieved through the application of large admissible perturbation (LEAP) theory. LEAP theory stems from the matrix perturbation techniques developed by Stetson[1] and improved by Stetson and Harrison,[2] Sandström and Anderson,[3] and Kim and Anderson.[4] These linear methods find an objective structure very close to the baseline structure and require a new finite element solution to proceed to the next increment. It is not required to rerun the finite element model using values found in the inadmissible linear prediction step in the implementation of LEAP theory. The LEAP theory approach[5–7] allows all incremental predictions to be made from the results of the baseline finite element solution and the corresponding nonlinear corrections. LEAP theory is applicable to a finite element representation of a single structure; the use of substructuring is not currently developed.

Assume a complex structural system has been decomposed into substructures (or that the substructure models have been developed independently) and that the substructures' dynamics have been cast in terms of eigensolutions. Of course, this implies that there is a truncated set of eigenvalues and eigenvectors that adequately describe the dynamics of each substructure. Component mode synthesis (CMS) allows the assembly of these modal substructure models to compose a structural system model. Among the first to contribute to this technique are Gladwell,[8] Hurty,[9] Craig and Bampton,[10] Goldman,[11] Hou,[12] Bajan et al.,[13] MacNeal,[14] and Benfield and Hruda.[15] One clear benefit of this approach is that a small number of modes can be used to describe the dynamics of the substructures for most applications.

Presently, the missing link between these two valuable techniques, LEAP theory and CMS, is developed. That is, a new redesign process that is applicable to a structural system model is developed. It is assumed that the baseline substructure models have been established during the initial design phase based on the basic geometry and material properties of each substructure. If some of the eigenvalues of the baseline structural system must be changed to meet some objective performance criteria, then an objective structural system model must be developed. The remainder of this work develops the process by which the baseline structural system is redesigned to form the objective structural system.

The objective of this work is to develop a method to predict the changes in the substructure eigenvalues that will affect the desired change in the structural system eigenvalues. This objective is achieved in three steps. The first step is the development of the first-order perturbation relationships, that relate changes in the eigenvalues of the substructures to the corresponding changes in the structural system eigenvalues. The development of these analytical perturbation relationships for structural systems is a major contribution of this work. Second, a transformation matrix is developed for each perturbation relationship, and these matrices are combined to form the ultimate structural transformation matrix. Finally, a minimization problem is defined that uses the ultimate transformation

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matrix as a set of constraint equations. The result of this minimization is a set of changes to the substructure eigenvalues that approximately affect the objective changes to the structural system eigenvalues.

II. Background: Component Mode Synthesis

CMS is used in this work to assemble the structural system. This technique allows substructures to be combined, via the substructure connectivity constraints, while retaining the analytical relationship between the substructure degrees of freedom and the structural system degrees of freedom. An outline of the technique is presented because this feature is consequential to the development of the new redesign process.

Assume each substructure is represented by second-order equations in the form

$$[m_{(r)}]\{\ddot{x}_{(r)}\} + [k_{(r)}]\{x_{(r)}\} = \{f_{(r)}\} \tag{1}$$

where $\{x_{(r)}\}$ is a vector that represents the physical degrees of freedom of the rth substructure. Note that damping can be included in Rayleigh form. The eigensolution is found based on a mixture of fixed and free geometric boundary conditions yielding a set of modes. These modes are normalized with respect to the mass matrix, thus forming the modal matrix $[\Phi^n_{(r)}]$. The modal substructure degrees of freedom $\{p_{(r)}\}$ are related to the physical substructure degrees of freedom by

$$\{x_{(r)}\} = [\Phi^n_{(r)}]\{p_{(r)}\} \tag{2}$$

Premultiplying Eq. (1) by $[\Phi^n_{(r)}]^T$ and substituting Eq. (2) yields

$$\{\ddot{p}_{(r)}\} + [\Lambda_{(r)}]\{p_{(r)}\} = \{0\} \tag{3}$$

where $[\Lambda_{(r)}]$ is the diagonal eigenvalue matrix of the rth substructure. These substructures are concatenated to form

$$\{x\} = \begin{Bmatrix} \{x_{(1)}\} \\ \{x_{(2)}\} \\ \vdots \end{Bmatrix} = \begin{bmatrix} [\Phi^n_{(1)}] & [0] & \cdots \\ [0] & [\Phi^n_{(2)}] & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{Bmatrix} \{p_{(1)}\} \\ \{p_{(2)}\} \\ \vdots \end{Bmatrix} = [\Phi^n]\{p\} \tag{4}$$

$$\{\ddot{p}\} = \begin{Bmatrix} \{\ddot{p}_{(1)}\} \\ \{\ddot{p}_{(2)}\} \\ \vdots \end{Bmatrix} = \left[\begin{array}{c|c|c} -[\Lambda_{(1)}] & [0] & \cdots \\ \hline [0] & -[\Lambda_{(2)}] & \cdots \\ \hline \vdots & \vdots & \ddots \end{array}\right] \begin{Bmatrix} \{p_{(1)}\} \\ \{p_{(2)}\} \\ \vdots \end{Bmatrix} = [A_{\text{sub}}]\{p\} \tag{5}$$

The first step in the synthesis process is to write the physical constraint equations, corresponding to the substructure connectivity, in terms of the physical degrees of freedom of the substructures, that is,

$$[A_{\text{constraint}}]\{x\} = \{0\} \tag{6}$$

The physical constraint equations are converted to modal degrees of freedom:

$$[A_{\text{constraint}}]\{x\} = [A_{\text{constraint}}][\Phi^n]\{p\} = \lfloor A_{\text{constraint},\Phi^n}\rfloor\{p\} = \{0\} \tag{7}$$

and partitioned into $[A_d]$ and $[A_g]$ such that $[A_d]$ is square and $\{p\}$ is partitioned into $\{p_d\}$ and $\{p_g\}$ to correspond with the partitioning of $[A_d]$ and $[A_g]$, that is,

$$[A_{\text{constraint},\Phi^n}]\{p\} = [[A_d][A_g]]\begin{Bmatrix}\{p_d\} \\ \{p_g\}\end{Bmatrix} = \{0\} \tag{8}$$

Now $\{p_d\}$ represents the dependent degrees of freedom, and $\{p_g\}$ represents the independent degrees of freedom. The relationship between the dependent and independent degrees of freedom is

$$\{p\} = \begin{Bmatrix}\{p_d\} \\ \{p_g\}\end{Bmatrix} = [\beta]\{p_g\} = [\beta]\{q\} \tag{9}$$

where

$$[\beta] = \left[\begin{array}{c} -[A_d]^{-1}[A_g] \\ \hline [I] \end{array}\right] \tag{10}$$

The degrees of freedom of the structural system $\{q\}$ are related to the modal degrees of freedom of the substructures $\{p\}$ via the $[\beta]$ matrix.

The $[\beta]$ transformation matrix is used to constrain the substructure equations to form the system equations:

$$\{\ddot{q}\} = ([\beta]^T[\beta])^{-1}[\beta]^T[A_{\text{sub}}][\beta]\{q\} = [A_{\text{sys}}]\{q\} \tag{11}$$

The eigenvalue problem for the structural system is

$$([A_{\text{sys}}] - \lambda_{\text{sys}}[I])\{\psi_{\text{sys}}\} = \{0\} \tag{12}$$

which can be solved for the eigenvalues and eigenvectors of the system, $\lambda_{\text{sys}\,i}$ and $\{\psi_{\text{sys}}\}_i$, and the eigenvalue and eigenvector matrices $[\Lambda_{\text{sys}}]$ and $[\Phi_{\text{sys}}] = [\{\psi_{\text{sys}}\}_1 \;\; \{\psi_{\text{sys}}\}_2 \;\; \ldots]$. The inverse of the eigenvector matrix is defined as

$$[\Gamma_{\text{sys}}] = [\Phi_{\text{sys}}]^{-1}, \qquad [\Gamma_{\text{sys}}]^T = [\{\gamma_{\text{sys}}\}_1 \;\; \{\gamma_{\text{sys}}\}_2 \;\; \ldots] \tag{13}$$

for computational convenience that will be demonstrated later.

III. Structural System Redesign

Assume that a baseline structural system model has been developed and that the eigenvalues are known. If any of the eigenvalues of the baseline structural system must be changed to meet some objective performance criteria, then an objective structural model must be developed. The objective of this work is to develop a process to predict the changes in the substructure eigenvalues that will affect the desired change in the structural system eigenvalues. This objective is achieved through the developments made in the following sections. The first-order perturbation relationships between changes in the structural system eigenvalues and changes in the substructure eigenvalues are developed in the first section. Next, the transformation matrices, which are based on the perturbation relationships developed in Sec. I, and the ultimate transformation matrix are developed. Finally, the ultimate transformation matrix is used to predict the changes in the substructure eigenvalues that will affect the objective changes in the structural system eigenvalues.

A. Developing Analytical Perturbation Relationships

The first step in the structural system redesign process is to examine the equations governing the eigenvalues for the objective structure. Let matrices representing the properties of the objective structure be represented with the primed notation. The equation for the diagonal eigenvalue matrix for the objective structure is

$$[\Lambda'_{\text{sys}}] = [\Gamma'_{\text{sys}}][A'_{\text{sys}}][\Phi'_{\text{sys}}] \tag{14}$$

or

$$([\Lambda_{\text{sys}}] + [\Delta\Lambda_{\text{sys}}]) = ([\Gamma_{\text{sys}}] + [\Delta\Gamma_{\text{sys}}])([A_{\text{sys}}] + [\Delta A_{\text{sys}}])([\Phi_{\text{sys}}] + [\Delta\Phi_{\text{sys}}]) \tag{15}$$

The matrix of admixture coefficients for $[\Phi'_{\text{sys}}]$ is $[C^{\Phi}]$, and the matrix of admixture coefficients for $[\Gamma_{\text{sys}}]$ is $[C^{\Gamma}]$, that is,

$$[\Phi'_{\text{sys}}] = [\Phi_{\text{sys}}] + [\Delta\Phi_{\text{sys}}] = [\Phi_{\text{sys}}] + [\Phi_{\text{sys}}][C^{\Phi}]^T \tag{16}$$

$$[\Gamma'_{\text{sys}}] = [\Gamma_{\text{sys}}] + [\Delta\Gamma_{\text{sys}}] = [\Gamma_{\text{sys}}] + [C^{\Gamma}][\Gamma_{\text{sys}}] \tag{17}$$

Substituting Eqs. (16) and (17) into Eq. (15) yields

$$([\Lambda_{\text{sys}}] + [\Delta\Lambda_{\text{sys}}]) = ([\Gamma_{\text{sys}}] + [C^{\Gamma}][\Gamma_{\text{sys}}])([A_{\text{sys}}] + [\Delta A_{\text{sys}}]) \times ([\Phi_{\text{sys}}] + [\Phi_{\text{sys}}][C^{\Phi}]^{\Gamma}) \tag{18}$$

Given that $[C^{\Phi}]$ is a first-order perturbation term, then $[C^{\Gamma}]$ is also a first-order term; this is developed in Proof 2 of Appendix A. The first-order perturbation of Eq. (18) is

$$[\Delta\Lambda_{\text{sys}}] = [C^{\Gamma}][\Lambda_{\text{sys}}] + [\Gamma_{\text{sys}}][\Delta A_{\text{sys}}][\Phi_{\text{sys}}] + [\Lambda_{\text{sys}}][C^{\Phi}]^T \tag{19}$$

and the diagonal terms are

$$\Delta\lambda_{\text{sys}_i} = \{\gamma_{\text{sys}}\}_i^T[\Delta A_{\text{sys}}]\{\Psi_{\text{sys}}\}_i \tag{20}$$

Note that the diagonal terms of $[C^\Gamma][\Lambda_{sys}]$ are of second order (for the conditions stated in Proof 2 of Appendix A) and that the diagonal terms of $[\Lambda_{sys}][C^\Phi]^T = 0$ and, therefore, that neither term is included in Eq. (20).

The first-order perturbation relationship between $[\Delta A_{sys}]$ and $[\Delta A_{sub}]$ and $[\Delta\beta]$, derived from Eq. (11), is

$$[\Delta A_{sys}] = \Delta(([\beta]^T[\beta])^{-1})[\beta]^T[A_{sub}][\beta] + ([\beta]^T[\beta])^{-1}[\Delta\beta]^T[A_{sub}][\beta] + ([\beta]^T[\beta])^{-1}[\beta]^T \times [\Delta A_{sub}][\beta] + ([\beta]^T[\beta])^{-1}[\beta]^T[A_{sub}][\Delta\beta] \quad (21)$$

Introducing the term $[C_{sys}] = [\beta]^T[\beta]$ provides a means of dividing the task of finding the first-order perturbation relationship into several smaller steps. With the introduction of this new matrix, the expression for the $[\Delta A_{sys}]$ relationship becomes

$$[\Delta A_{sys}] = \Delta([C_{sys}]^{-1})[\beta]^T[A_{sub}][\beta] + [C_{sys}]^{-1}[\Delta\beta]^T[A_{sub}][\beta] + [C_{sys}]^{-1}[\beta]^T[\Delta A_{sub}][\beta] + [C_{sys}]^{-1}[\beta]^T[A_{sub}][\Delta\beta] \quad (22)$$

Given the conditions stated in Proof 3 of Appendix A, the first-order perturbation of $\Delta([C_{sys}]^{-1})$ is

$$\Delta([C_{sys}]^{-1}) = -[C_{sys}]^{-1}[\Delta C_{sys}][C_{sys}]^{-1} \quad (23)$$

By definition, the $[C_{sys}]$ matrix is $[\beta]^T[\beta]$; therefore, the first-order perturbation of the $[C_{sys}]$ matrix is

$$[\Delta C_{sys}] = [\Delta\beta]^T[\beta] + [\beta]^T[\Delta\beta] \quad (24)$$

Now, the first-order perturbation relations are known between $[\Delta\lambda_{sys}]$ and $[\Delta\beta]$ and $[\Delta A_{sub}]$.

As shown in Proof 3 of Appendix A, the first-order perturbation of Eq. (10) is

$$[\Delta\beta] = \begin{bmatrix} [A_d]^{-1}[\Delta A_d][A_d]^{-1}[A_g] - [A_d]^{-1}[\Delta A_g] \\ \hdashline [0] \end{bmatrix} \quad (25)$$

Because $[A_{constraint}]$ is a constant matrix, for a given substructure connectivity, the first-order perturbation relation for $[A_{constraint}][\Phi^n]$ is

$$[\Delta A_{constraint\,\Phi^n}] = [A_{constraint}][\Delta\Phi^n] \quad (26)$$

These first-order perturbation relationships form the basis for the transformation matrices developed in step two of the structural system redesign process.

B. Developing Transformation Matrices

The transformation matrices that describe the changes in the structural system eigenvalues in terms of the changes in the substructure eigenvalues are developed in step two. The first transformation matrix is formed by examining Eq. (20), which is the basis for the redesign equation for the ith system eigenvalue. Assume that not all of the system eigenvalues require a prescribed change. For instance, if the first, third, and seventh eigenvalues are to be changed, then

$$\{\Delta\lambda_{sys}\} = \begin{Bmatrix} \Delta\lambda_{sys_1} \\ \Delta\lambda_{sys_3} \\ \Delta\lambda_{sys_7} \end{Bmatrix} \quad (27)$$

In this case, the first, third, and seventh system eigenvalues will define the redesign constraint equations [not to be confused with the physical constraint equations of Eq. (6)].

Step two begins by writing Eq. (20) in the form of a transformation matrix for all of the system eigenvalues that define the redesign constraint equations. This transformation matrix $[T_{\lambda\,sys,A\,sys}]$ describes the first-order perturbation relationships between changes in $[A_{sys}]$ and changes in $\{\lambda_{sys}\}$, that is,

$$\{\Delta\lambda_{sys}\} = [T_{\lambda_{sys},A_{sys}}]\{\Delta A_{sys}\} \quad (28)$$

The transformation matrix predicting changes in $[A_{sys}]$ based on changes on $[C_{sys}]^{-1}$, $[\beta]$, and $[A_{sub}]$, developed from Eq. (22), is

$$\{\Delta A_{sys}\} = [T_{A_{sys},C_{sys}^{-1}}]\{\Delta(C_{sys})^{-1}\} + [T_{A_{sys},\beta}]\{\Delta\beta\} + [T_{A_{sys},A_{sub}}]\{\Delta A_{sub}\} \quad (29)$$

The transformation matrix from the $[C_{sys}]$ space to the $[C_{sys}]^{-1}$ space, following directly from Eq. (23), is

$$\{\Delta(C_{sys})^{-1}\} = [T_{C_{sys}^{-1},C_{sys}}]\{\Delta C_{sys}\} \quad (30)$$

The transformation matrix for changes in $[C_{sys}]$ due to changes in $[\beta]$, taken from Eq. (24), is then

$$\{\Delta C_{sys}\} = [T_{C_{sys},\beta}]\{\Delta\beta\} \quad (31)$$

At this point, several first-order perturbation relationships have been derived and the corresponding transformation matrices have been developed. Equations (28–31) are combined to form

$$\{\Delta\lambda_{sys}\} = [T_{\lambda_{sys},\beta}]\{\Delta\beta\} + [T_{\lambda_{sys},A_{sub}}]\{\Delta A_{sub}\} \quad (32)$$

where

$$\lfloor T_{\lambda_{sys},\beta}\rfloor = (\lfloor T_{\lambda_{sys},A_{sys}}\rfloor\lfloor T_{A_{sys},C_{sys}^{-1}}\rfloor\lfloor T_{C_{sys}^{-1},C_{sys}}\rfloor\lfloor T_{C_{sys},\beta}\rfloor + \lfloor T_{\lambda_{sys},A_{sys}}\rfloor\lfloor T_{A_{sys},\beta}\rfloor) \quad (33)$$

$$\lfloor T_{\lambda_{sys},A_{sub}}\rfloor = \lfloor T_{\lambda_{sys},A_{sys}}\rfloor\lfloor T_{A_{sys},A_{sub}}\rfloor \quad (34)$$

The formation of transformation matrices continues with the change in $[\beta]$ arising from changes in the physical constraint equations, developed from Eq. (25), written as

$$\{\Delta\beta\} = \lfloor T_{\beta,A_{constraint\,\Phi}}\rfloor\{\Delta A_{constraint\,\Phi^n}\} \quad (35)$$

The conversion from the changes in substructure eigenvectors to changes in physical constraints, derived from Eq. (26), is

$$\{\Delta A_{constraint\,\Phi^n}\} = [T_{A_{constraint\,\Phi},\Phi}]\{\Delta\Phi^n\} \quad (36)$$

The transformations described by Eqs. (35) and (36) are combined with Eq. (32) to form

$$\{\Delta\lambda_{sys}\} = [T_{\lambda_{sys},\Phi}]\{\Delta\Phi^n\} + [T_{\lambda_{sys},A_{sub}}] + \{\Delta A_{sub}\} \quad (37)$$

where

$$\lfloor T_{\lambda_{sys},\Phi}\rfloor = \lfloor T_{\lambda_{sys},\beta}\rfloor\lfloor T_{\beta,A_{constraint\,\Phi}}\rfloor\lfloor T_{A_{constraint\,\Phi},\Phi}\rfloor \quad (38)$$

Next, $\{\Delta\Phi^n\}$ and $\{\Delta A_{sub}\}$ will be related back to $\{\Delta\lambda_{sub}\}$.

It is assumed that the sensitivity of the substructure eigenvectors with respect to the substructure eigenvalues is known a priori. This step is consequential to the development of the perturbation relationships and can be a limiting factor in the magnitude of the allowable changes in the system eigenvalues. The sensitivity can be determined easily, however, via a numerical sensitivity study or the application of LEAP theory. Because small changes in the substructure eigenvalues are being made, a numerical sensitivity study may be more straightforward and appropriate. Note that the sensitivity relationship for each substructure can be determined independently so that these relationships can be processed in parallel. The sensitivity relationship is represented, in transformation matrix form, as

$$\{\Delta\Phi^n\} = [T_{\Phi,\lambda_{sub}}]\{\Delta\lambda_{sub}\} \quad (39)$$

It is clear from Eq. (5) that $[A_{sub}]$ was created from the substructure eigenvalues. The transformation matrix between $\{\Delta\lambda_{sub}\}$ and $\{\Delta A_{sub}\}$, therefore, is composed of zeros and ones,

$$\{\Delta A_{sub}\} = [T_{A_{sub},\lambda_{sub}}]\{\Delta\lambda_{sub}\} \quad (40)$$

The transformations in Eqs. (39) and (40) are combined with those found in Eq. (37) to form the ultimate transformation matrix

$$\{\Delta\lambda_{\text{sys}}\} = [T_{\lambda_{\text{sys}},\lambda_{\text{sub}}}]\{\Delta\lambda_{\text{sub}}\} \tag{41}$$

where

$$[T_{\lambda_{\text{sys}},\lambda_{\text{sub}}}] = [T_{\lambda_{\text{sys}},A_{\text{sys}}}]\begin{pmatrix}([T_{A_{\text{sys}},C_{\text{sys}}^{-1}}][T_{C_{\text{sys}}^{-1},C_{\text{sys}}}][T_{C_{\text{sys}},\beta}] + [T_{A_{\text{sys}},\beta}])[T_{\beta,A_{\text{constraint}\,\Phi}}][T_{A_{\text{constraint}\,\Phi},\Phi}][T_{\Phi,\lambda_{\text{sub}}}] \\ + [T_{A_{\text{sys}},A_{\text{sub}}}][T_{A_{\text{sub}},\lambda_{\text{sub}}}]\end{pmatrix} \tag{42}$$

The ultimate structural transformation matrix [Eq. (42)] results from deriving the first-order perturbation relationship between the substructure eigenvalues and the structural system eigenvalues. This transformation matrix is the basis for the third step in the structural redesign process.

C. Predicting Substructure Changes

The third step in the structural system redesign process is completed by solving a simple minimization problem. The structural transformation matrix developed in step two is a set of equality constraint equations. The objective is to minimize the sum of the square of the fractional change in the substructure eigenvalues while satisfying the constraints. A scaling technique must be applied to Eq. (41), therefore, before this objective function can be used. This scaling will facilitate minimizing the fractional changes in the eigenvalues instead of the absolute changes.

The α notation is used to denote a fractional change, and the $\{\alpha_{\lambda\,\text{sys}}\}$ and $\{\alpha_{\lambda\,\text{sub}}\}$ vectors are defined such that

$$\{\Delta\lambda_{\text{sys}}\} \equiv \lfloor \Lambda_{\text{sys}} \rfloor \{\alpha_{\lambda\,\text{sys}}\}, \qquad \{\Delta\lambda_{\text{sub}}\} \equiv [\Lambda_{\text{sub}}]\{\alpha_{\lambda\,\text{sub}}\} \tag{43}$$

A scaled transformation matrix $[T_\alpha]$ is formed such that

$$\{\alpha_{\lambda_{\text{sys}}}\} = [T_\alpha]\{\alpha_{\lambda_{\text{sub}}}\} \tag{44}$$

where

$$[T_\alpha] = [1/\Lambda_{\text{sys}}][T_{\lambda_{\text{sys}},\lambda_{\text{sub}}}][\Lambda_{\text{sub}}] \tag{45}$$

and the constraint equations are now written in terms of fractional changes.

This minimization problem may be satisfied via the generalized inverse when the problem is underconstrained. The solution to the underconstrained minimization problem are the predicted fractional changes in the substructure eigenvalues:

$$\{\alpha^*_{\lambda\,\text{sub}}\} = [T_\alpha]^+\{\alpha_{\lambda\,\text{sys}}\} \tag{46}$$

where

$$[T_\alpha]^+ = [T_\alpha]^T([T_\alpha][T_\alpha]^T)^{-1} \tag{47}$$

or

$$\{\Delta\lambda^*_{\text{sub}}\} = [\Lambda_{\text{sub}}][T_\alpha]^+[1/\Lambda_{\text{sys}}]\{\Delta\lambda_{\text{sys}}\} \tag{48}$$

The result of the structural system redesign process is a set of changes to the substructure eigenvalues that approaches the objective changes to the structural system eigenvalues. Specifically, the objective function of this minimization problem is the sum of the squared fractional changes of the substructure eigenvalues. One feature of this objective function makes it desirable for the structural system redesign process. The burden of satisfying the redesign constraint equations for the objective structural system eigenvalues is distributed among all of the fractional changes in the substructure eigenvalues, independent of the magnitude of those substructure eigenvalues. This allows relatively large changes in the structural system eigenvalues compared to other objective functions.

IV. Example

Two simple types of substructures are used to demonstrate the structural system redesign process. These suspension and plate substructures are represented schematically in Fig. 1. The dynamic equations for these substructures are in modal form, the plate was derived via a finite element analysis, and the suspension was derived from first principles. The specific numerical values are listed in Appendix B and summarized in Table A1.

The structural system is an isolated platform, shown schematically in Fig. 2, consisting of a plate and four suspension substructures. The plate is substructure 1 and includes rigid body modes and three elastic modes. Only the eigenvalues corresponding to the elastic modes can be changed, so that only these eigenvalues are included in the transformation matrices. The four suspensions are substructures 2–5. Substructures 2–4 include two modes whereas substructure 5 includes one mode. The structural system is synthesized via CMS. The first five eigenvalues for this structure are: 18.36, 34.16, 39.35, 72.67, and 183.01 rad/s.

It is assumed that the first two structural system eigenvalues must be changed. Knowing which eigenvalues are to be changed, along with the mathematical description of the baseline structure, is all of the information needed to begin the redesign process. Each transformation matrix in the structural system redesign process is formed from the corresponding first-order perturbation relationship. The $[T_\alpha]$ transformation matrix for this example is

$$[T_\alpha] = \left[\begin{array}{ccc|cc|cc|cc|c} -2.72E+1 & 3.55E-0 & 7.31E+1 & 1.27E-1 & -1.10E-0 & 6.92E-0 & 2.35E-0 & 8.50E-2 & -8.59E-1 & 5.29E-2 \\ -1.90E-2 & -2.27E-2 & -5.26E-2 & 3.33E-1 & -3.03E-3 & 4.55E-0 & -3.81E-8 & 2.12E-8 & -2.06E-7 & 6.68E-1 \end{array}\right] \tag{49}$$

This transformation is used to predict the changes in the substructure eigenvalues that will produce, in turn, the objective changes in the structural system eigenvalues.

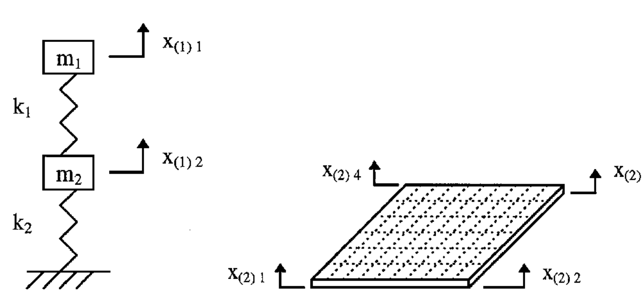

Fig. 1 Schematic of suspension and plate substructures.

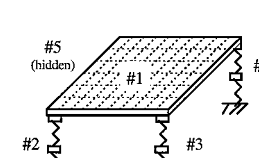

Fig. 2 Schematic of an isolated platform.

Each set of objective changes in the structural system eigenvalues corresponds to a predicted set of changes in the substructure eigenvalues. For instance, when the objective change in the first eigenvalue of the structure is -2% and the objective change in the second eigenvalue is $+1\%$, then the corresponding fractional changes in the substructure eigenvalues are

$$\{\alpha_{\lambda_{\text{sub}}}\} = [T_\alpha]^+\{\alpha_{\lambda_{\text{sys}}}\} = \left\{\begin{matrix} \begin{Bmatrix} -5.15E-4 \\ -3.82E-4 \\ -4.68E-4 \end{Bmatrix} \\ \begin{Bmatrix} 5.96E-3 \\ -6.14E-5 \end{Bmatrix} \\ \begin{Bmatrix} 4.48E-6 \\ 1.52E-5 \end{Bmatrix} \\ \begin{Bmatrix} 5.51E-7 \\ -5.57E-6 \end{Bmatrix} \\ \{1.19E-2\} \end{matrix}\right\} \tag{50}$$

Introducing the predicted substructure eigenvalue changes in the full nonlinear equations, in general, will not exactly produce the objective changes in the structural system eigenvalues. The structural system eigenvalues that result from this substitution are called the actual eigenvalues. The actual eigenvalues and objective eigenvalues of the structural system are, respectively,

$$\{\lambda_{\text{sys}}\}_{\text{actual}} = \begin{Bmatrix} 17.97 \\ 34.55 \end{Bmatrix}, \qquad \{\lambda_{\text{sys}}\}_{\text{objective}} = \begin{Bmatrix} 17.99 \\ 34.50 \end{Bmatrix} \tag{51}$$

The error between the ith objective system eigenvalue and the ith actual eigenvalue is defined as

$$(\lambda_{\text{sys}\,i})_{\text{error}} = \frac{(\lambda_{\text{sys}\,i})_{\text{objective}} - (\lambda_{\text{sys}\,i})_{\text{actual}}}{(\lambda_{\text{sys}\,i})_{\text{objective}}} \tag{52}$$

and the error in the first and second structural system eigenvalues are

$$(\lambda_{\text{sys}\,1})_{\text{error}} = 0.12\%, \qquad (\lambda_{\text{sys}\,2})_{\text{error}} = -0.13\% \tag{53}$$

This error is small with respect to the objective changes as expected.

The value of the actual structural system eigenvalues is dependent on all of the objective eigenvalue changes. That is, the actual structural system eigenvalues are derived from the predicted changes in the substructure eigenvalues. The predicted changes in the substructure eigenvalues are determined, in turn, by the objective changes in all of the structural system eigenvalues via Eq. (41). The error for each of the structural system eigenvalues, as defined by Eq. (52), is, therefore, a function of the objective changes in all of the structural system eigenvalues. Presently, this will become clear when a domain of structural system eigenvalue changes is introduced.

Changes in the first and second eigenvalues of the structural system produce a two-dimensional domain. The error between the objective eigenvalues and the resulting eigenvalues is calculated over this domain via Eq. (52). The result is an error surface over the domain of the changes in objective eigenvalues. Because there are two eigenvalues being changed, two error surfaces will result. The first error surface corresponds to the error in the first eigenvalue over the domain, and the second error surface corresponds to the error in the second eigenvalue. The first and second eigenvalues are varied by $\pm 5\%$ and the error surfaces resulting from this domain are shown in Figs. 3 and 4.

V. Discussion

The applicability of the process needs to be addressed. The structural system may be represented by modal expansions but not necessarily represented by finite element models. In fact, the redesign process developed in this work is applicable to any assemblage of substructures that can be represented by linear, time-invariant sets of equations. In this work, however, the development of the redesign process is presented in terms of a structural description.

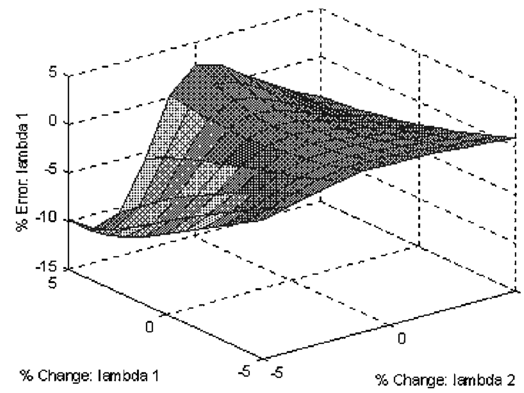

Fig. 3 Error surface for the first eigenvalue of the structural system.

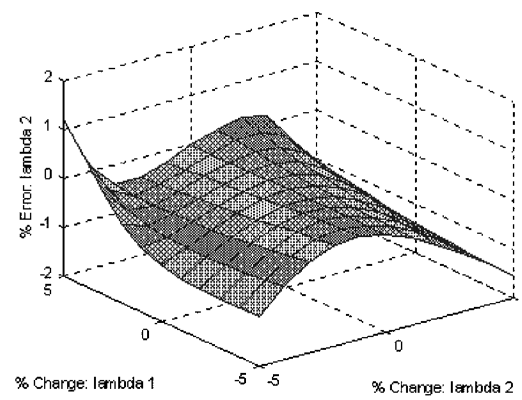

Fig. 4 Error surface for the second eigenvalue of the structural system.

The results summarized in Figs. 3 and 4 demonstrate two important properties of the structural system redesign process. First, the accuracy of the process can be evaluated for this example. Recall that the transformation matrices are derived from first-order perturbation relationships. The error surfaces, therefore, should exhibit two properties of first-order perturbation relationships. The error in eigenvalue predictions should be zero when the objective change in the eigenvalues is zero, and the slope at this zero–zero point should be zero. Both Figs. 3 and 4 exhibit these properties. The application of this process, therefore, is yielding appropriate results. The second issue is the domain of allowable changes for this example.

Appropriate application of this process is dependent on the magnitude of the objective changes in the structural system eigenvalues. This directly results from the assumptions made in the proofs presented in Appendix A. In the example, the changes shown for the first and second eigenvalues are $\pm 5\%$. Figure 4 shows that this domain is appropriate according to the error in the second eigenvalue. The maximum error in the second eigenvalue is -1.6%; this error occurs when the first eigenvalue is changed by -5% and the second eigenvalue is changed by $+5\%$. Part of this domain, however, is not appropriate according to the error in the first eigenvalue. Figure 3 shows that the error in the first eigenvalue is large (i.e., on the order of the objective changes in the eigenvalues) when the change in the first eigenvalue is positive and the change in the second eigenvalue is negative. To achieve changes in this direction, several smaller incremental changes must be made.

In addition to the issues raised by the example, several development and implementation issues exist with this process and need to be addressed. The development issue is the use of multiple transformation matrices. This representation has two major advantages. The major advantage is that it reduces an otherwise complex set of perturbation equations into several smaller sets of perturbation equations. This makes the derivation and explanation of the ultimate transformation matrix easier. Perhaps more importantly, it provides a framework on which future perturbation relationships can be derived more readily.

The first-order perturbation relationship derived relates the changes in the structural system eigenvalues to the changes in the substructure eigenvalues. In future work, it may be desirable to find the perturbation relationships involving other changes. These changes may include global unforced responses such as structural system modes or forced responses such as maximum displacements, accelerations, or stresses. If other changes are desired, the first-order perturbation relationship needs only to be found with respect to a perturbation relationship that is already known. For instance, the relationship between changes in the structural modes and the substructure eigenvalues may be desired. Because changes in the structural modes $[\Phi_{\text{sys}}]$ are functions of the changes in $[A_{\text{sys}}]$, the $[T_{\Phi\,\text{sys},A\,\text{sys}}]$ transformation matrix would have to be derived. This would be the only new transformation matrix that would have to be derived. The $[T_{A\,\text{sys},\lambda\,\text{sub}}]$ transformation matrix is already presented in this work. In this way, less development work will be required as various first-order perturbation relationships are developed in the future.

A majority of the first-order perturbation relationships developed are required due to the $[\beta]$ transformation matrix. It is tempting to assume that first-order changes in the substructure eigenvectors only cause second-order changes in the system eigenvalues. The system eigenvalues are based on Ritz approximations of the system eigenvectors that are linear combinations of the substructure eigenvectors, and, therefore, first-order changes to the substructure eigenvectors should only cause second-order changes to the system eigenvalues. This is not the case, however, due to the matrix inversions required in the formation of $[A_{\text{sys}}]$ and $[\beta]$. The Ritz argument is applicable when $[A_{\text{sys}}]$ is constant and the Ritz functions approximating the system eigenvectors are perturbed. This explicit second-order relationship is accounted for in Eq. (6) when the diagonal terms are eliminated. In this case, however, $[A_{\text{sys}}]$ itself is being perturbed.

VI. Conclusions

The process developed in this work is a powerful new engineering tool for redesigning structural systems. This work establishes the critical link between two authoritative modeling techniques, component mode synthesis and large admissible perturbation theory. For the first time, analytical perturbation results are developed for the redesign of a structural system composed of a virtually unlimited hierarchy of substructures. This will allow new flexibility in the redesign of structural models. Because the perturbation relationships are cast in terms of transformation matrices, this process forms the basis for future developments in structural system redesign when other static and dynamic objectives are to be achieved.

Appendix A: Proofs

The following definitions are required for the ensuing proofs:

1) Here, ε is an arbitrarily small positive number.

2) Here, n is the number of rows and columns in the following matrices.

3) $[X]$ is an arbitrary invertable baseline matrix.

4) $[Y]$ is defined as $[X]^{-1}$.

5) $[X']$ and $[Y']$ are invertable objective matrices.

6) $[\Delta X]$ and $[\Delta Y]$ are matrices of changes to the elements of $[X]$ and $[Y]$, respectively.

7) $[C^X]$ and $[C^Y]$ are admixture coefficient matrices, where $C^x_{ii} = 0$, $|C^x_{ij}| < \varepsilon$.

8) Here, $(\,)_{ij}$ refers to the element in row i and column j of the matrix within $(\,)$.

Then

$$[X'] \equiv [X] + [\Delta X] \equiv [X] + [X][C^X] \equiv [X]([I] + [C^X])$$

$$[Y'] \equiv [Y] + [\Delta Y] \equiv [Y] + [C^Y][Y] \equiv ([I] + [C^Y])[Y]$$

$$[Y] \equiv [X]^{-1}, \qquad [Y'] \equiv [X']^{-1}$$

Proof 1: We wish to prove that the error in the estimate $([I] + [C^X])^{-1} \cong ([I] - [C^X])$ is bounded by some small $\delta \ll \varepsilon$ when the admixture coefficients for the matrix $[X]$ are bounded by ε. That is, we wish to prove that every term in the matrix $(([I] + [C^X])^{-1} - ([I] - [C^X]))$ is bounded by δ, or $|(([I] + [C^X])^{-1} - ([I] - [C^X]))_{ij}| < \delta$, where $\delta \ll \varepsilon$.

Premultiplying the identity $[I] - [C^X][C^X] \equiv ([I] + [C^X])([I] - [C^X])$ by $([I] + [C^X])^{-1}$ and postmultiplying by $([I] - [C^X][C^X])^{-1}$ yields

$$([I] + [C^X])^{-1} = ([I] - [C^X])([I] - [C^X]^2)^{-1}$$

Note that, in general,

$$[I] - [C^X]^{2m} \equiv ([I] - [C^X]^m)([I] + [C^X]^m)$$

or

$$([I] - [C^X]^m)^{-1} = ([I] + [C^X]^m)([I] - [C^X]^{2m})^{-1}$$

This leads to

$$([I] + [C^X])^{-1} = ([I] - [C^X])([I] + [C^X]^2)([I] + [C^X]^4)([I] + [C^X]^8)\ldots$$

or

$$([I] + [C^X])^{-1} = ([I] - [C^X])([I] + [C^X]^2) \times ([I] + [C^X]^4)([I] + [C^X]^8)\ldots$$

which yields

$$([I] + [C^X])^{-1} = ([I] - [C^X])\left([I] + \sum_{m=1}^{\infty} [C^X]^{2m}\right)$$

and subtracting $([I] - [C^X])$ gives

$$([I] + [C^X])^{-1} - ([I] - [C^X]) = [C^X]^2 \sum_{m=0}^{\infty} (-1)^m [C^X]^m$$

Recall that $|C^X_{ij}| < \varepsilon$, $\forall i \neq j$, and so $|([C^X]^2)_{ij}| < n\varepsilon^2$. Then the terms in the matrix $(([I] + [C^X])^{-1} - ([I] - [C^X]))$ are bounded by

$$\left|(([I] + [C^X])^{-1} - ([I] - [C^X]))_{ij}\right| < n\varepsilon^2 \sum_{m=0}^{\infty} (n\varepsilon)^m = \frac{n\varepsilon^2}{1 - n\varepsilon}$$

leading directly to

$$\left|(([I] + [C^X])^{-1} - ([I] - [C^X]))_{ij}\right| < \delta$$

where $\delta = [n\varepsilon^2/(1 - n\varepsilon)]$ and $\delta \ll \varepsilon$ for values of $\varepsilon \ll 1/2n$.

Proof 2: We wish to prove that the error in the estimate $[C^Y] \cong -[C^X]$ is bounded by some small $\delta \ll \varepsilon$ when the admixture coefficients for the matrix $[X]$ are bounded by ε. A new matrix $[D]$ is defined as $[D] \equiv [C^Y] + [C^X]$. Now we wish to prove that every term in the matrix $[D]$ is bounded by $\delta \ll \varepsilon$, or $|D_{ij}| < \delta$, where $\delta \ll \varepsilon$.

$[Y']$ is defined such that

$$[Y'][X'] \equiv [I]$$

expanding primed matrices

$$([Y] + [\Delta Y])([X] + [\Delta X]) = [I]$$

and subtracting $[Y][X] \equiv [I]$,

$$[Y][\Delta X] + [\Delta Y]([X] + [\Delta X]) = [0]$$

Substituting the matrix of admixture coefficients for $[\Delta X]$ and $[\Delta Y]$ yields

$$[Y][X][C^X] + [C^Y][Y]([X] + [X][C^X]) = [0]$$

or

$$[C^X] + [C^Y]([I] + [C^X]) = [0]$$

and substituting $[D] \equiv [C^Y] + [C^X]$,

$$[D] = [C^X][C^X] - [D][C^X]$$

or

$$[D]([I] + [C^X]) = [C^X][C^X]$$

Postmultiplying by $([I] + [C^X])^{-1}$ yields an expression for $[D]$:

$$[D] = [C^X]^2([I] + [C^X])^{-1}$$

where each term is

$$D_{ij} = \sum_{k=1}^{n}([C^X]^2)_{ik}(([I] + [C^X])^{-1})_{kj}$$

It is known from Proof 1 that $|(([I] + [C^X])^{-1} - ([I] - [C^X]))_{ij}| < \delta$, where $\delta = n\varepsilon^2/(1 - n\varepsilon)$, and recall that $|C^X_{ij}| < \varepsilon$, so that $|([C^X]^2)_{ij}| < n\varepsilon^2$. Then the terms of $[D]$ are bounded by

$$|D_{ij}| < \sum_{k=1}^{n}(n\varepsilon^2)_{ik}\left(([I] - [C^X])_{kj} + \frac{n\varepsilon^2}{1 - n\varepsilon}\right)$$

or

$$|D_{ij}| < n\varepsilon^2 + n^2\varepsilon^3 + [n^3\varepsilon^4/(1 - n\varepsilon)]$$

and the proof that $|D_{ij}| < \delta$, where $\delta = \{n\varepsilon + n^2\varepsilon^2 + [n^3\varepsilon^3/(1 - n\varepsilon)]\}\varepsilon$ and $\delta \ll \varepsilon$ for values of $\varepsilon \ll 1/2n$.

Proof 3: We wish to prove that the error in the estimate $\Delta([X]^{-1}) \cong -[X]^{-1}[\Delta X][X]^{-1}$, or, equivalently, $[\Delta Y] \cong -[Y][\Delta X][Y]$ is bounded by some small $\delta \ll \varepsilon$ when the admixture coefficients for the matrix $[X]$ are bounded by ε and the terms of $[Y]$ are all bounded by ξ, that is, $|Y_{ij}| < \xi$. First a new matrix $[D]$ is defined as $[D] \equiv [Y][\Delta X][Y] + [\Delta Y]$. That is, we wish to prove that every term in the matrix $[D]$ is bounded by d, or $|D_{ij}| < \delta$, where $\delta \ll \varepsilon$.

Recall that $[Y']$ is defined such that $[X'][Y'] = [I]$. Expanding the prime terms,

$$([X] + [\Delta X])([Y] + [\Delta Y]) = [I]$$

and subtracting $[X][Y] = [I]$, we have

$$[\Delta X][Y] + ([X] + [\Delta X])[\Delta Y] = [0]$$

Substituting the admixture coefficient matrices for $[\Delta X]$ and $[\Delta Y]$ yields

$$[\Delta X][Y] + [X]([I] + [C^X])[C^Y][Y] = [0]$$

and premultiplying by $[X]^{-1} = [Y]$ gives

$$[Y][\Delta X][Y] + [\Delta Y] + [C^X][C^Y][Y] = [0]$$

or

$$[D] = -[C^X][C^Y][Y]$$

Now from Proof 2 it follows that

$$\left|([C^X][C^Y])_{ij}\right| < 2n\varepsilon^2$$

for values of $\varepsilon < 1/2n$. Because the terms of $[Y]$ are all bounded by ξ, then $|D_{ij}| < \delta$ where $\delta = 2n^2\varepsilon^2\xi$, and $\delta \ll \varepsilon$ for values of $\varepsilon \ll 1/2n^2\xi$.

Appendix B: Example Details

Plate Substructure

The plate substructure (Fig. A1) is 600 mm long and has 6 plate elements in the x direction and is 1000 mm long and has 10 plate elements in the y direction. The thickness of the plate elements varies as indicated hereafter, where the thickness values are in millimeters. The modulus of elasticity is 206,800 N/(mm)^2, Poisson's ratio is 0.3,

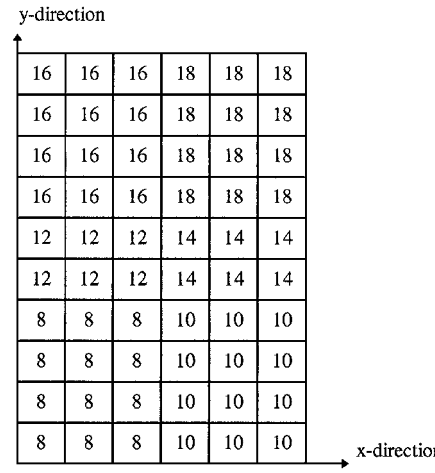

Fig. A1 Plate substructure.

and the density is $7.82E-9$ kg/(mm)^3. The first six eigenvalues and eigenvectors (for the displacements at the four corners) are

$$\lambda_1 = \lambda_2 = \lambda_3 = 0, \qquad \lambda_4 = 1.536362E{+}05$$

$$\lambda_5 = 1.800375E{+}05, \qquad \lambda_6 = 9.821307E{+}05$$

$$[\Phi] = \begin{bmatrix} -7.389 & 5.075 & 7.799 & 13.777 & 18.338 & -20.455 \\ 4.557 & -1.923 & 10.246 & 8.498 & -16.696 & 10.432 \\ -8.941 & -2.404 & -4.540 & 5.061 & -7.344 & -10.034 \\ 3.005 & -9.402 & -2.093 & 5.932 & 5.594 & 9.140 \end{bmatrix}$$

Suspension Substructures

The suspension substructures are given in Table A1.

Table A1 Suspension substructures

Substructure number	$m1$	$m2$	$k1$	$k2$	λ_1	$\{\psi_1\}$	λ_2	$\{\psi_2\}$
2	20	10	1000	2500	34.169	0.218	365.83	−0.049
						0.069		0.309
3	40	20	1000	2500	17.084	0.154	182.92	−0.034
						0.049		0.218
4	20	10	2000	5000	68.338	0.218	731.66	−0.049
						0.069		0.309
5	40	20	2000	5000	34.169	0.154	365.83	−0.034
						0.049		0.218

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